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ABSTRACT

Geometric algebra plays a major role in merging the physical and mathematical ideas in the context of various physical systems. In this paper, we explore certain properties associated with barotropic and non-barotropic fluid flows with the help of geometric algebra over a four-dimensional Euclidean space time manifold. We introduce the concepts of multivectors associated with vorticity, helicity, and parity, which evolve from a four-velocity field. In this context, the fluid dynamical analogs of the Poynting theorem, Lorentz force, and Maxwell's equations are derived. The *fluid Maxwell's equations* can be extracted from a single equation.

I. INTRODUCTION

Mathematical frameworks for representing dynamical systems include differential and geometric algebra in addition to differential and integral calculus. Vector notation is the most commonly used tool to describe physical systems, but it is restricted to the three dimensional space. In higher dimensions, advanced analytical tools such as that of differential forms, tensors and spinors play a significant role in complex fields of special relativity, quantum mechanics, and string theory.

William Clifford introduced geometric algebra during the second half of 19th century by combining the algebraic ideas of Hamilton and Grassmann. Geometric algebra was structured by combining the outer and inner products of elements of a linear space into a new product called geometric product. The inclusion of geometrical concepts into abstract Clifford algebra has enriched geometric algebra and has made it evolve into a powerful mathematical tool. The introduction of geometric algebra into classical mechanics was done by Hestenes through his work "Space Time Algebra" in 1966. Through a series of papers, Hestenes advocated the applications of geometric algebra in wide range of fields such as quantum mechanics, electrodynamics, relativistic physics, Lie group theory, and projective and conformal geometry.[1,2] Detailed description of the subject of geometric algebra can also be seen in the works of Doran, Lasenby, and Lasenby[3] and Bromborsky.[4] However, its applications in the field of fluid dynamics have been restricted to specialized problems like that of fluid flows with variable viscosity in quaternionic settings, elliptic boundary value problems, and visualization of vector fields in the study of gas dynamics and combustion. A geometric algebraic approach to fluid dynamics to derive Kelvin's circulation theorem and Helmholtz' vorticity theorems can be seen in the work of Cibura and Hildenbrand.[5]

Here, we would like to bridge the gap by extending the field of applications of geometric algebra to derive certain quantities such as multivectors associated with vorticity, helicity, and parity in barotropic and non-barotropic flows. The paper is structured as follows: the basic concepts of geometric algebra over a three dimensional Euclidean space time manifold are introduced in Secs. II and III. Section IV gives an overview of governing equations of both barotropic and non-barotropic flows and Sec. V provides expressions for velocity, vorticity, helicity, and parity in the language of differential forms. A four-velocity is introduced in Sec. VI. In

Secs. VII, VIII, and IX, the properties of the geometric product are used to define multivectors associated with vorticity, helicity, and parity. In Sec. X, a fluid dynamic analog of the four dimensional stress–energy tensor is derived, which in turn leads to the Poynting theorem and Lorentz force. We also find that the *fluid Maxwell's equations* can be retrieved from a single equation [Eq. (41)].

II. GEOMETRIC ALGEBRA

Let V be a finite dimensional vector space over real $\mathbb{R}$. V together with a geometric product is called a geometric algebra $\mathcal{G}$. For $a, b \in V$, this geometric product is defined as

$$ab = a \cdot b + a \wedge b,$$

where $a \cdot b$ is an inner product and $a \wedge b$ is termed the outer product (exterior product). Here, the inner product is the usual scalar product. The outer product of two vectors a and b, $(a \wedge b)$, is defined to be a *bivector* and is taken as the signed area of a parallelogram with a and b as its sides. The outer product of three vectors $a \wedge b \wedge c$ (*trivector*) is taken as an oriented volume of a parallelepiped with the three vectors forming its sides. Generalizing this, the outer product of n vectors is visualized to be an n dimensional parallelogram with a magnitude and an orientation associated with it.

The outer product of k vectors is called a *blade of grade-k*. Scalars are grade-0 blades, vectors and their linear combination are grade-1 blades, bivectors and their linear combination are grade-2 blades, and so on. For an n dimensional vector space, grade-n blades are called *pseudoscalars* and grade-(n-1) blades are called *pseudovectors*. Thus, the geometric product is a sum of the scalar product and the outer product and is termed a *multivector*. Hence, the elements of a geometric algebra are multivectors. These multivectors are linear combinations of blades of different grades. The sum and products of multivectors are also multivectors so that $\mathcal{G}$ is algebraically closed.

III. SPACE TIME ALGEBRA

Space time algebra (STA) is the geometric algebra of space and time and is commonly referred by physicists to as the Minkowski space time. The geometric algebra $G(4, 0)$ or $\mathcal{G}_4$ of the Euclidean four dimensional space R^4 and the geometric algebra $\mathcal{G}(1,3)$ of Minkowski space time $R^{1,3}$ are found to be algebraically isomorphic.[6] Here, we consider the geometric algebra of the Euclidean space time manifold only.

Let $\{e_0, e_1, e_2, e_3\}$ be a set of orthonormal vectors tangent to a point lying in the Euclidean space time manifold. e_o is taken to be the time like vector and e_1, e_2, e_3 are the space vectors. The e_i's satisfy the relations $e_i^2 = 1$, $e_i \cdot e_j = \delta_{ij}$, the Kronecker delta where i and j take values from 0 to 3. The six bivectors $e_i \wedge e_j$ satisfy $(e_i \wedge e_j)^2 = -1$ and they generate rotations in corresponding planes. The four trivectors satisfy $e_i e_j e_k = \epsilon_{ijkl} I e_l$, where I is the grade-4 pseudoscalar defined as $I = e_0 e_1 e_2 e_3$ and can be considered as the unit four dimensional oriented volume element. The trivector, or otherwise termed pseudovector, can be considered as a three dimensional unit volume. Thus, the set $\{e_i\}$ generates a basis consisting of 16 terms (1 scalar, 4 vectors, 6 bivectors, 4 trivectors, and 1 pseudoscalar) for the geometric algebra $\mathcal{G}_4$ of the four dimensional Euclidean space R^4.

IV. GOVERNING EQUATIONS FOR FLUID FLOWS

A. Barotropic flows in $\mathbb{R}^3$

Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be a divergence free velocity field in $\mathbb{R}^3$. The dynamics of an ideal incompressible flow with potential body force is described by the Euler equation,

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - \nabla\varphi, \tag{1}$$

and that of viscous incompressible fluids are described by the Navier–Stokes' equation,

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - \nabla\varphi + \nu\nabla^2\boldsymbol{v}, \tag{2}$$

where p is the pressure, ρ is the mass density, φ is the potential body force, and ν is the kinematic viscosity.[7] The identity $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla\frac{\boldsymbol{v}^2}{2} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})$ is used to convert the above equations into the form

$$\partial_t \boldsymbol{v} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla\varepsilon \tag{3}$$

and

$$\partial_t \boldsymbol{v} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla\varepsilon + \nu\nabla^2\boldsymbol{v}, \tag{4}$$

respectively, where $\varepsilon = \frac{\boldsymbol{v}^2}{2} + \frac{p}{\rho} + \varphi$ is the Bernoulli function.

In terms of the divergence free vorticity field $\boldsymbol{w} = \nabla \times \boldsymbol{v}$, the vorticity equation for the inviscid incompressible flow is given by

$$\partial_t \boldsymbol{w} - \nabla \times (\boldsymbol{v} \times \boldsymbol{w}) = 0, \tag{5}$$

and for viscous flow,

$$\partial_t \boldsymbol{w} - \nabla \times (\boldsymbol{v} \times \boldsymbol{w}) = \nu\nabla^2\boldsymbol{w}. \tag{6}$$

Taking the dot products of (3) and (4) with velocity, we get

$$\partial_t\left(\frac{\boldsymbol{v}^2}{2}\right) + \boldsymbol{v} \cdot \nabla\varepsilon = 0$$

and

$$\partial_t\left(\frac{\boldsymbol{v}^2}{2}\right) + \boldsymbol{v} \cdot \nabla\varepsilon = \boldsymbol{v} \cdot \nu\nabla^2\boldsymbol{v}.$$

B. Non-barotropic flows in $\mathbb{R}^3$

A flow is said to be non-barotropic when the pressure does not depend on density alone but on another quantity such as specific entropy or, in certain cases, chemical composition.[8] We consider non-barotropic flows for which $p = p(\rho, S)$, where S is the specific entropy.

Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be velocity field in $\mathbb{R}^3$. The equations describing non-barotropic flows are given as follows:

(1) equation of continuity

$$\partial_t \rho + \nabla \cdot (\rho\boldsymbol{v}) = 0, \tag{7}$$

(2) conservation of momentum

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla(I+\varphi) + T\nabla S \tag{8}$$

for perfect fluids and

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla(I+\varphi) + T\nabla S + \nu\nabla^2\boldsymbol{v} + \frac{\nu}{3}\nabla(\nabla\cdot\boldsymbol{v}) \tag{9}$$

for viscous fluids,[9] and

(3) conservation of entropy

$$\partial_t S + (\boldsymbol{v}\cdot\nabla)S = 0. \tag{10}$$

Here, we have used the relation

$$-\nabla I + T\nabla S = -\nabla p/\rho, \tag{11}$$

where I is the specific enthalpy and T is the absolute temperature.

Defining the generalized velocity $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \tau\nabla S$, where τ is the thermasy and using the relation

$$\frac{D(\tau\nabla S)}{Dt} = T\nabla S - \tau[(\nabla S\cdot\nabla)\boldsymbol{v} + (\nabla S\times\boldsymbol{w})]$$

as given in Ref. 10, Eq. (8) takes the form

$$\begin{aligned}\partial_t\tilde{\boldsymbol{v}} - \boldsymbol{v}\times\tilde{\boldsymbol{w}} &= -\nabla\left(\frac{\boldsymbol{v}^2}{2} + I + \varphi - \boldsymbol{v}\cdot\tau\nabla S\right)\\ &= \nabla\tilde{f}\end{aligned} \tag{12}$$

for non-barotropic perfect flows and Eq. (9) takes the form

$$\begin{aligned}\partial_t\tilde{\boldsymbol{v}} - \boldsymbol{v}\times\tilde{\boldsymbol{w}} &= -\nabla\left(\frac{\boldsymbol{v}^2}{2} + I + \varphi - \boldsymbol{v}\cdot\tau\nabla S\right) + \nu\nabla^2\boldsymbol{v} + \frac{\nu}{3}\nabla(\nabla\cdot\boldsymbol{v})\\ &= \nabla\tilde{f} + \nu\nabla^2\boldsymbol{v} + \frac{\nu}{3}\nabla(\nabla\cdot\boldsymbol{v})\end{aligned} \tag{13}$$

for viscous flows, where $\tilde{f} = -(\frac{\boldsymbol{v}^2}{2} + I + \varphi - \boldsymbol{v}\cdot\tau\nabla S)$, which we define as the generalized Bernoulli function. Following Mobbs, we define the generalized vorticity as $\tilde{\boldsymbol{w}} = (\boldsymbol{w} - \nabla\tau\times\nabla S) = \nabla\times\tilde{\boldsymbol{v}}$.

V. DIFFERENTIAL FORMS ASSOCIATED WITH BAROTROPIC FLOWS

A. One form of velocity

Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be a divergence-free velocity field in $\mathbb{R}^3$ satisfying Euler's Eq. (1) or Navier–Stokes' Eq. (2). Associated with the four-velocity vector field $\boldsymbol{\mathcal{V}} = (f, \boldsymbol{v})$ in $\mathbb{R}^4$, then the velocity one form is expressed as

$$\sigma = f dt + v_1 dx_1 + v_2 dx_2 + v_3 dx_3.$$

In particular, by choosing $f = -\varepsilon$, that is, the negative of the Bernoulli function,[11] then the velocity one form is expressed as

$$\sigma = -\varepsilon dt + v_1 dx_1 + v_2 dx_2 + v_3 dx_3. \tag{14}$$

B. Two form of vorticity

From the velocity one form σ as in (14), the vorticity two form is defined as $\omega = d\sigma$

$$\omega = (\boldsymbol{v}\times\boldsymbol{w})dt\wedge d\mathbf{x} + \boldsymbol{w}\cdot d\boldsymbol{s} \tag{15}$$

for ideal flows and

$$\omega = ((\boldsymbol{v}\times\boldsymbol{w}) + \nu\nabla^2\boldsymbol{v})dt\wedge d\mathbf{x} + \boldsymbol{w}\cdot d\boldsymbol{s} \tag{16}$$

for viscous flows. $\boldsymbol{w} = \nabla\times\boldsymbol{v} = (w_1, w_2, w_3)$ is the vorticity vector in $\mathbb{R}^3$. In $\mathbb{R}^4$, the vorticity two form can also be represented as $(\boldsymbol{L}, \boldsymbol{w})$, where $\boldsymbol{L} = \partial_t\boldsymbol{v} + \nabla\varepsilon = (L_1, L_2, L_3)$. Here, $\boldsymbol{L} = \boldsymbol{v}\times\boldsymbol{w}$ the negative of the Lamb vector for ideal fluids and $\boldsymbol{L} = (\boldsymbol{v}\times\boldsymbol{w}) + \nu\nabla^2\boldsymbol{v}$ for viscous fluids.

C. Three form of helicity

From the velocity one form σ and the vorticity two form ω, we can define

$$\eta = \sigma\wedge\omega$$

as the three form associated with helicity. Then, for Euler flows, we have

$$\eta = \boldsymbol{v}\cdot\boldsymbol{w}dx_1\wedge dx_2\wedge dx_3 + dt\wedge((\boldsymbol{v}\times\boldsymbol{w})\times\boldsymbol{v} - \varepsilon\boldsymbol{w})\cdot d\boldsymbol{s}, \tag{17}$$

and for viscous flows, we have

$$\eta = \boldsymbol{v}\cdot\boldsymbol{w}dx_1\wedge dx_2\wedge dx_3 + dt\wedge(((\boldsymbol{v}\times\boldsymbol{w}) + \nu\nabla^2\boldsymbol{v})\times\boldsymbol{v} - \varepsilon\boldsymbol{w})\cdot d\boldsymbol{s}. \tag{18}$$

Hence, $\eta = h dx_1\wedge dx_2\wedge dx_3 - dt\wedge H\cdot d\mathbf{s}$, where $h = \boldsymbol{v}\cdot\boldsymbol{w}$ is the helicity density and $-H = (\partial_t\boldsymbol{v} + \nabla\varepsilon)\times\boldsymbol{v} - \varepsilon\boldsymbol{w}$.

D. Four form of parity

From the vorticity two form ω, one could define the four form parity as

$$\kappa = \omega\wedge\omega. \tag{19}$$

The four form κ vanishes for ideal flows. For viscous flows, we have

$$\begin{aligned}\kappa &= 2\nu\nabla^2\boldsymbol{v}\cdot\boldsymbol{w}dt\wedge dx_1\wedge dx_2\wedge dx_3\\ &= -2\nu(\nabla\times\boldsymbol{w})\cdot\boldsymbol{w}dt\wedge dx_1\wedge dx_2\wedge dx_3.\end{aligned} \tag{20}$$

VI. GEOMETRIC ALGEBRAIC APPROACH: FOUR-VELOCITY FIELDS IN FLUID FLOWS

In this section, we apply geometric algebraic methods to fluid dynamics by introducing four-velocity fields for barotropic and non-barotropic flows.

A. Barotropic flows

The four-velocity field of a barotropic fluid flow $\boldsymbol{\mathcal{V}} = (f, \boldsymbol{v})$ can be expressed as a vector in $\mathcal{G}_4$ as

$$\boldsymbol{V} = fe_0 + v_1 e_1 + v_2 e_2 + v_3 e_3,$$

along with the incompressibility condition or the divergence free condition

$$\nabla\cdot\boldsymbol{V} = \partial_t f + \partial_{x_1}v_1 + \partial_{x_2}v_2 + \partial_{x_3}v_3 = 0,$$

where ∇ denotes the vector derivative,

$$\nabla = e_0 \frac{\partial}{\partial t} + \nabla.$$

For barotropic flows, one should replace f with the negative of the Bernoulli function ε ($f = -\varepsilon$).

B. Non-barotropic flows

For non-barotropic flows, a generalized four velocity, $\tilde{\mathcal{V}} = (\tilde{f}, \tilde{\boldsymbol{v}})$, can be expressed as a vector in $\mathcal{G}_4$ as

$$\tilde{\boldsymbol{V}} = \tilde{f} e_0 + \tilde{v}_1 e_1 + \tilde{v}_2 e_2 + \tilde{v}_3 e_3.$$

VII. BIVECTOR ASSOCIATED WITH VORTICITY

A. Barotropic flows

For the given velocity $\boldsymbol{V}$, the derivative of the velocity can be expressed as a bivector

$$\begin{aligned} \boldsymbol{W} &= \nabla \boldsymbol{V} = \nabla \cdot \boldsymbol{V} + \nabla \wedge \boldsymbol{V} \\ &= (\partial_t v_1 + \partial_{x_1} \varepsilon) e_0 \wedge e_1 + (\partial_t v_2 + \partial_{x_2} \varepsilon) e_0 \wedge e_2 \\ &\quad + (\partial_t v_3 + \partial_{x_3} \varepsilon) e_0 \wedge e_3 + w_1 e_2 \wedge e_3 + w_2 e_3 \wedge e_1 + w_3 e_1 \wedge e_2, \end{aligned} \tag{21}$$

where $\boldsymbol{w}$ is vorticity vector. Here, $\boldsymbol{W}$ is a grade-2 blade associated with two forms of the generalized vorticity. Hence, the vorticity field takes the form of an oriented surface. Thus, for ideal and viscous flows, we have $\boldsymbol{W}$ as

$$\begin{aligned} \boldsymbol{W} &= L_1 e_0 \wedge e_1 + L_2 e_0 \wedge e_2 + L_3 e_0 \wedge e_3 + w_1 e_2 \wedge e_3 \\ &\quad + w_2 e_3 \wedge e_1 + w_3 e_1 \wedge e_2. \end{aligned}$$

B. Non-barotropic flows

For the given generalized velocity $\tilde{\boldsymbol{V}}$, the derivative of the velocity can be expressed as a bivector

$$\begin{aligned} \tilde{\boldsymbol{W}} &= \nabla \tilde{\boldsymbol{V}} = \nabla \cdot \tilde{\boldsymbol{V}} + \nabla \wedge \tilde{\boldsymbol{V}} \\ &= (\partial_t \tilde{f} + \partial_{x_1} \tilde{v}_1 + \partial_{x_2} \tilde{v}_2 + \partial_{x_3} \tilde{v}_3) + (\partial_t \tilde{v}_1 - \partial_{x_1} \tilde{f}) e_0 \wedge e_1 \\ &\quad + (\partial_t \tilde{v}_2 - \partial_{x_2} \tilde{f}) e_0 \wedge e_2 + (\partial_t \tilde{v}_3 - \partial_{x_3} \tilde{f}) e_0 \wedge e_3 + \tilde{w}_1 e_2 \wedge e_3 \\ &\quad + \tilde{w}_2 e_3 \wedge e_1 + \tilde{w}_3 e_1 \wedge e_2, \end{aligned}$$

where $\tilde{\boldsymbol{w}} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ is the three dimensional generalized vorticity vector. $\tilde{\boldsymbol{W}}$ is thus the sum of a scalar and a grade-2 blade. This grade-2 blade of generalized vorticity vector is associated with the two form of the generalized vorticity for non-barotropic flows.[12]

Defining $\tilde{\boldsymbol{L}} = \boldsymbol{v} \times \tilde{\boldsymbol{w}}$, we can rewrite the $\tilde{\boldsymbol{W}}$ as follows:

$$\begin{aligned} \tilde{\boldsymbol{W}} &= (\partial_t \tilde{f} + \partial_{x_1} \tilde{v}_1 + \partial_{x_2} \tilde{v}_2 + \partial_{x_3} \tilde{v}_3) + \tilde{L}_1 e_0 \wedge e_1 + \tilde{L}_2 e_0 \wedge e_2 \\ &\quad + \tilde{L}_3 e_0 \wedge e_3 + \tilde{w}_1 e_2 \wedge e_3 + \tilde{w}_2 e_3 \wedge e_1 + \tilde{w}_3 e_1 \wedge e_2. \end{aligned}$$

VIII. MULTIVECTOR ASSOCIATED WITH HELICITY

A. Barotropic flows

The geometric product of velocity vector $\boldsymbol{V}$ and the vorticity bivector $\boldsymbol{W}$ gives a multivector, which can be associated with the helicity. Thus,

$$\boldsymbol{H} = \boldsymbol{V}\boldsymbol{W} = \boldsymbol{V} \cdot \boldsymbol{W} + \boldsymbol{V} \wedge \boldsymbol{W}.$$

Here,

$$\begin{aligned} \boldsymbol{V} \cdot \boldsymbol{W} &= -(\boldsymbol{v} \cdot \boldsymbol{L}) e_0 + (-\varepsilon L_1 + (\boldsymbol{w} \times \boldsymbol{v})_1) e_1 + (-\varepsilon L_2 + (\boldsymbol{w} \times \boldsymbol{v})_2) e_2 \\ &\quad + (-\varepsilon L_3 + (\boldsymbol{w} \times \boldsymbol{v})_3) e_3 \\ &= [-(\boldsymbol{v} \cdot \mathbf{L}), -\varepsilon \mathbf{L} + (\boldsymbol{w} \times \boldsymbol{v})] \\ &= (-h', -\hat{H}'). \end{aligned} \tag{22}$$

The quantity $-h' = -(\mathbf{v} \cdot \mathbf{L})$ vanishes for ideal fluids, but not for viscous fluids. Hence, for ideal fluids, $\boldsymbol{V} \cdot \boldsymbol{W}$ is a usual vector in the three dimensional space. Here, $-\hat{H}' = -\varepsilon L + (w \times v) = (-H_1', -H_2', -H_3')$. Now,

$$\begin{aligned} \boldsymbol{V} \wedge \boldsymbol{W} &= (\boldsymbol{v} \cdot \boldsymbol{w}) e_1 e_2 e_3 + (-\varepsilon w_1 + (\boldsymbol{L} \times \boldsymbol{v})_1) e_0 e_2 e_3 \\ &\quad + (-\varepsilon w_2 + (\boldsymbol{L} \times \boldsymbol{v})_2) e_0 e_3 e_1 + (-\varepsilon w_3 + (\boldsymbol{L} \times \boldsymbol{v})_3) e_0 e_1 e_2 \\ &= I[-(\boldsymbol{v} \cdot \boldsymbol{w}), -\varepsilon \boldsymbol{w} + (\boldsymbol{L} \times \boldsymbol{v})] \\ &= I(-h, -\hat{H}), \end{aligned} \tag{23}$$

where $h = \boldsymbol{v} \cdot \boldsymbol{w}$ is the helicity density and $-\hat{H} = -\varepsilon \boldsymbol{w} + (\boldsymbol{L} \times \boldsymbol{v}) = (-H_1, -H_2, -H_3)$. Therefore, $\boldsymbol{H}$ is the sum of a vector and a grade-3 blade. The dual of the trivector part $\boldsymbol{H}_T$ of $\boldsymbol{H}$ gives a four-vector ($-h$, $-\hat{H}$), which can be associated with the charge density four-vector. The trivector part of $\boldsymbol{H}$ can be associated with the helicity three-form, as given in Eq. (17). The vector part of the above geometric product contains additional terms whose physical and geometrical relevance need to be further harnessed.

B. Non-barotropic flows

As in the case of barotropic flows, the geometric product of generalized velocity vector $\tilde{\boldsymbol{V}}$ and the vorticity bivector $\tilde{\boldsymbol{W}}$ gives a multivector that can be associated with the helicity of non-barotropic flows. Thus,

$$\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{V}}\tilde{\boldsymbol{W}} = \tilde{\boldsymbol{V}} \cdot \tilde{\boldsymbol{W}} + \tilde{\boldsymbol{V}} \wedge \tilde{\boldsymbol{W}},$$

$$\begin{aligned} \tilde{\boldsymbol{V}}\tilde{\boldsymbol{W}} &= 2\tilde{\boldsymbol{V}} \cdot (\nabla \cdot \tilde{\boldsymbol{V}}) - \tilde{\boldsymbol{v}}.(\boldsymbol{v} \times \tilde{\boldsymbol{w}}) e_0 + (\tilde{f}\tilde{L}_1 - (\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{w}})_1) e_1 \\ &\quad + (\tilde{f}\tilde{L}_2 - (\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{w}})_2) e_2 + (\tilde{f}\tilde{L}_3 - (\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{w}})_3) e_3 \\ &\quad + (\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{w}}) e_1 \wedge e_2 \wedge e_3 + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_1 e_0 \wedge e_2 \wedge e_3 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_2 e_0 \wedge e_3 \wedge e_1 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_3 e_0 \wedge e_1 \wedge e_2. \\ &= 2\tilde{\boldsymbol{V}} \cdot (\nabla \cdot \tilde{\boldsymbol{V}}) + [-\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{L}}, \tilde{f}\tilde{\boldsymbol{L}} - (\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{w}})] + (\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{w}}) e_1 \wedge e_2 \wedge e_3 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_1 e_0 \wedge e_2 \wedge e_3 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_2 e_0 \wedge e_3 \wedge e_1 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + (\partial_t \tilde{\boldsymbol{v}} - \nabla \tilde{f}) \times \tilde{\boldsymbol{v}})_3 e_0 \wedge e_1 \wedge e_2. \\ &= 2\tilde{\boldsymbol{V}} \cdot (\nabla \cdot \tilde{\boldsymbol{V}}) + [-\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{L}}, \tilde{f}\tilde{\boldsymbol{L}} - (\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{w}})] + (\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{w}}) e_1 \wedge e_2 \wedge e_3 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{L}} \times \tilde{\boldsymbol{v}})_1 e_0 \wedge e_2 \wedge e_3 + (\tilde{f}\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{L}} \times \tilde{\boldsymbol{v}})_2 e_0 \wedge e_3 \wedge e_1 \\ &\quad + (\tilde{f}\tilde{\boldsymbol{w}} + \tilde{\boldsymbol{L}} \times \tilde{\boldsymbol{v}})_3 e_0 \wedge e_1 \wedge e_2 \end{aligned} \tag{24}$$

is the sum of a vector and a trivector. The trivector part of this geometric product gives the generalized helicity for non-barotropic flows,[12] while the scalar and the vector quantities obtained above are new and their significance needs to be further studied.

IX. MULTIVECTOR ASSOCIATED WITH PARITY

A. Barotropic flows

Evaluating $\boldsymbol{W}^2 = \boldsymbol{W}\boldsymbol{W}$, we get

$$\boldsymbol{W}^2 = \boldsymbol{W} \cdot \boldsymbol{W} + \boldsymbol{W} \wedge \boldsymbol{W}.$$

Here, the scalar part is given by

$$\boldsymbol{W} \cdot \boldsymbol{W} = -(L_1^2 + L_2^2 + L_3^3 + w_1^2 + w_2^2 + w_3^2) = -(|\boldsymbol{L}|^2 + |\boldsymbol{w}|^2),$$

and the pseudoscalar part is evaluated as

$$\boldsymbol{W} \wedge \boldsymbol{W} = 2(\boldsymbol{L} \cdot \boldsymbol{w})e_0 \wedge e_1 \wedge e_2 \wedge e_3.$$

Thus,

$$\boldsymbol{W}\boldsymbol{W} = -(|\boldsymbol{L}|^2 + |\boldsymbol{w}|^2) + 2(\boldsymbol{L} \cdot \boldsymbol{w})e_0 \wedge e_1 \wedge e_2 \wedge e_3, \tag{25}$$

which is the sum of scalar and pseudoscalar. The pseudoscalar part of Eq. (25) can be associated with the four form of parity, as given in Eq. (19).

B. Non-barotropic flows

Evaluating $\tilde{\boldsymbol{W}}^2 = \tilde{\boldsymbol{W}}\tilde{\boldsymbol{W}}$, we get

$$\begin{aligned}\tilde{\boldsymbol{W}}^2 &= \tilde{\boldsymbol{W}} \cdot \tilde{\boldsymbol{W}} + \tilde{\boldsymbol{W}} \wedge \tilde{\boldsymbol{W}}, \qquad (26)\\
&= -(\tilde{L}_1^2 + \tilde{L}_2^2 + \tilde{L}_3^3 + \tilde{w}_1^2 + \tilde{w}_2^2 + \tilde{w}_3^2) + 2(\nabla \cdot \tilde{\boldsymbol{V}})(\nabla \cdot \tilde{\boldsymbol{V}})\\
&\quad + 2(\tilde{\boldsymbol{L}} \cdot \tilde{\boldsymbol{w}})e_0e_1e_2e_3 + 4(\nabla \cdot \tilde{\boldsymbol{V}})\big[\tilde{L}_1 e_0 \wedge e_1 + \tilde{L}_2 e_0 \wedge e_2\\
&\quad + \tilde{L}_3 e_0 \wedge e_3 + \tilde{w}_1 e_2 \wedge e_3 + \tilde{w}_2 e_3 \wedge e_1 + \tilde{w}_3 e_1 \wedge e_2\big]\\
&= -(|\tilde{\boldsymbol{L}}|^2 + |\tilde{\boldsymbol{w}}|^2) + 2(\nabla \cdot \tilde{\boldsymbol{V}})(\nabla \cdot \tilde{\boldsymbol{V}}) + 2(\tilde{\boldsymbol{L}} \cdot \tilde{\boldsymbol{w}})e_0e_1e_2e_3\\
&\quad + 4(\nabla \cdot \tilde{\boldsymbol{V}})\big[\tilde{L}_1 e_0 \wedge e_1 + \tilde{L}_2 e_0 \wedge e_2 + \tilde{L}_3 e_0 \wedge e_3 + \tilde{w}_1 e_2 \wedge e_3\\
&\quad + \tilde{w}_2 e_3 \wedge e_1 + \tilde{w}_3 e_1 \wedge e_2\big], \qquad (27)\end{aligned}$$

which is the sum of a scalar, bivector, and a pseudoscalar. The pseudoscalar part of this geometric product gives the generalized parity for non-barotropic flows.[12]

X. POYNTING THEOREM, LORENTZ FORCE, AND MAXWELL'S EQUATIONS IN FLUID DYNAMICS

An analogy between Maxwell stress tensor in electromagnetic theory and a tensor associated with vorticity in incompressible inviscid fluids was explored by Vedan *et al.*[13] For an incompressible three dimensional flow of an ideal fluid, the vorticity field is frozen-in and satisfies the vorticity Eq. (5). Analogous to the magnetic energy, the energy associated with the vorticity can be defined as $\dfrac{|\boldsymbol{w}|^2}{2} = \dfrac{\boldsymbol{w} \cdot \boldsymbol{w}}{2}$.

The rate of change of this energy can be expressed as $\partial_t\left(\dfrac{|\boldsymbol{w}|^2}{2}\right)$. Suppose that the vorticity is confined to a sub domain of the fluid, then by integrating the above equation over volume V of the sub domain, we get

$$\frac{dM}{dt} = \int_V \partial_t\left(\frac{|\boldsymbol{w}|^2}{2}\right)dV = -\int_V \mathbf{v} \cdot \mathbf{P}dV, \tag{28}$$

where $\mathbf{P} = (\nabla \times \boldsymbol{w}) \times \boldsymbol{w}$.

Here,

$$M = \int_V \left(\frac{|\boldsymbol{w}|^2}{2}\right)dV$$

is the total enstrophy of the volume V. Then, the ith component of $\mathbf{P}$, i.e., $P_i = \partial_{x_j}\Pi_{ij}$, where $\Pi_{ij} = w_i w_j - \dfrac{|\boldsymbol{w}|^2}{2}\delta_{ij}$, is the ijth entry of the matrix $\Pi = [\Pi_{ij}]$,

$$\Pi = \begin{bmatrix} w_1^2 - \dfrac{|\boldsymbol{w}|^2}{2} & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 - \dfrac{|\boldsymbol{w}|^2}{2} & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 - \dfrac{|\boldsymbol{w}|^2}{2} \end{bmatrix}. \tag{29}$$

The vorticity stress tensor Π thus obtained is associated with the enstrophy in the same way as the magnetic stress tensor is associated with the magnetic energy. The normal components of this tensor represent the tensile stress in the vortices and the tangential components represent the shearing forces on the plane to which $\boldsymbol{w}$ is normal, i.e., the plane associated with $\boldsymbol{w}$ as a bivector. The expressions of the vorticity stress tensor can be compared with that of the magnetic stress tensor discussed by Stierstadt and Liu.[14] Here, they point out that even though Maxwell's stress tensor is well known, its correct application to practical problems is not recognized even among experts and present a survey of the electromagnetic stress tensor and the electromagnetic forces in strongly polarizable materials. Whether the tensile stress tensor derived above can be related to the vortex stretching needs to be investigated.

The vorticity two form in a three dimensional manifold is expressed as $\omega = w_1 dx_2 \wedge dx_3 + w_2 dx_3 \wedge dx_1 + w_3 dx_1 \wedge dx_2$. We have

$$W = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}$$

as the tensor associated with the vorticity two form. The dual of the vorticity two form ω is a one form $*\omega = w_1 dx_1 + w_2 dx_2 + w_3 dx_3$, where $*$ denotes the Hodge dual operator. For an n dimensional manifold equipped with an Euclidean metric, the Hodge dual operator is a linear map from a k form $dx_{i_1} \wedge dx_{i_2} \ldots . \wedge dx_{i_k}$ to an $(n-k)$ form given by

$$*(dx_{i_1} \wedge dx_{i_2} \ldots . \wedge dx_{i_k}) = (-1)^m (dx_{i_{k+1}} \wedge dx_{i_{k+2}} \ldots . \wedge dx_{i_n}),$$

where $m = \begin{cases} 0 & \text{if } i_1, i_2, \ldots i_n \text{ is an even permutation of } \{1, 2, \ldots n\}, \\ 1 & \text{otherwise} \end{cases}$

and $i_{k+1} < i_{k+2} < \cdots < i_n$.

Let W^* denote the (3×1) column matrix associated with $*\omega$. The vorticity stress tensor can be derived from W by taking

$$\Pi = \frac{1}{2}\left(WW + W^* W^{*T}\right), \tag{30}$$

where W^{*T} is the transpose of W^*. The two form of vorticity

$$\omega = L_1 dt \wedge dx_1 + L_2 dt \wedge dx_2 + L_3 dt \wedge dx_3 + w_1 dx_2 \wedge dx_3 + w_2 dx_3 \wedge dx_1 + w_3 dx_1 \wedge dx_2$$

given in Eq. (15) in the four dimensional case can be associated with a vorticity field tensor F in $\mathbb{R}^4$. This tensor can be expressed as

$$F = \begin{bmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & w_3 & -w_2 \\ -L_2 & -w_3 & 0 & w_1 \\ -L_3 & w_2 & -w_1 & 0 \end{bmatrix}.$$

Let ∇' denote the matrix $\left[\partial_t \partial_{x_1} \partial_{x_2} \partial_{x_3}\right]$. Let $q = \nabla^2 \varepsilon$ and $\mathbf{J} = (J_1, J_2, J_3) = -\partial_t^2 \mathbf{v} - \partial_t \nabla \varepsilon + \nabla \times \boldsymbol{w}$. The matrix product $\nabla' F$ results in the equations

$$\begin{aligned} -\partial_{x_1} L_1 - \partial_{x_2} L_2 - \partial_{x_3} L_3 &= -q, \\ \partial_t L_1 - \partial_{x_2} w_3 + \partial_{x_3} w_2 &= -J_1, \\ \partial_t L_2 + \partial_{x_1} w_3 - \partial_{x_3} w_1 &= -J_2, \\ \partial_t L_3 - \partial_{x_1} w_2 + \partial_{x_2} w_1 &= -J_3, \end{aligned}$$

which can be expressed as $\nabla' F = \left[-q - J_1 - J_2 - J_3\right]$. The above equations are

$$\nabla \cdot \mathbf{L} = q \tag{31}$$

and

$$-\partial_t \mathbf{L} + \nabla \times \boldsymbol{w} = \mathbf{J}. \tag{32}$$

The dual of the two form ω is given by

$$*\omega = w_1 dt \wedge dx_1 + w_2 dt \wedge dx_2 + w_3 dt \wedge dx_3 + L_1 dx_2 \wedge dx_3 + L_2 dx_3 \wedge dx_1 + L_3 dx_1 \wedge dx_2.$$

$*\omega$ can be associated with the tensor F^* given by

$$F^* = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ -w_1 & 0 & L_3 & -L_2 \\ -w_2 & -L_3 & 0 & L_1 \\ -w_3 & L_2 & -L_1 & 0 \end{bmatrix}.$$

Evaluating $\nabla' F^*$, we get the equations

$$\begin{aligned} -\partial_{x_1} w_1 - \partial_{x_2} w_2 - \partial_{x_3} w_3 &= 0, \\ \partial_t w_1 - \partial_{x_2} L_3 + \partial_{x_3} L_2 &= 0, \\ \partial_t w_2 + \partial_{x_1} L_3 - \partial_{x_3} L_1 &= 0, \\ \partial_t w_3 - \partial_{x_1} L_2 + \partial_{x_2} L_1 &= 0, \end{aligned}$$

which in vector notation can be expressed as

$$\nabla \cdot \boldsymbol{w} = 0 \tag{33}$$

and

$$\partial_t \boldsymbol{w} - \nabla \times \mathbf{L} = 0. \tag{34}$$

Thus, we have $\nabla' F^* = \left[0\ 0\ 0\ 0\right]$.

Equations (31)–(34) together form the equations analogous to Maxwell's equations in electromagnetic theory formulated in 1865. The formulations of equations in fluid dynamics analogous to Maxwell's equations can be seen in the works of Marmanis,[15] Kambe,[16] and Sridhar.[17]

Equations (31)–(34) form a set of fluid Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{w} &= 0, \\ \nabla \times \mathbf{L} &= \partial_t \boldsymbol{w}, \\ \nabla \cdot \mathbf{L} &= q, \\ \nabla \times \boldsymbol{w} &= \mathbf{J} + \partial_t \mathbf{L}. \end{aligned}$$

Here, $q = \nabla^2 \varepsilon$ is termed the *hydrodynamic charge density* and $\mathbf{J} = -\partial_t^2 \mathbf{v} - \partial_t \nabla \varepsilon + \nabla \times \boldsymbol{w}$ is termed the *hydrodynamic current vector*. Marmanis recognized the vorticity and the Lamb vector as the kernel of a dynamical theory of turbulence and introduced the concept of *turbulent charge* and *turbulent current* in the study of meta fluid dynamics.[15] A study of the effectiveness of the Lamb vector and the hydrodynamic charge density in analyzing the dynamics of coherent structures of turbulent flow can be seen in Rousseaux *et al.*[18] The first two equations in the above set can be considered as the conservation and evolution equations for the vorticity and the last two equations as the conservation and the evolution of the Lamb vector. Also, we have $\det F = \det F^* = (\mathbf{L} \cdot \boldsymbol{w})^2$. Here, F is analogous to the electromagnetic tensor $\mathcal{F}$ in electromagnetic theory. We have

$$d\omega = (\nabla \cdot \boldsymbol{w}) dx_1 \wedge dx_2 \wedge dx_3 + (\partial_t \boldsymbol{w} - \nabla \times \mathbf{L}) dt \wedge ds$$

and

$$d(*\omega) = (\nabla \cdot \mathbf{L}) dx_1 \wedge dx_2 \wedge dx_3 + (\partial_t \mathbf{L} - \nabla \times \boldsymbol{w}) dt \wedge ds$$

or

$$*d(*\omega) = -(\nabla \cdot \mathbf{L}) dt + (\partial_t \mathbf{L} - \nabla \times \boldsymbol{w}) \cdot d\boldsymbol{x}.$$

Thus, the above set of four fluid Maxwell's equations can be extracted from the vorticity two form ω using the two equations

$$d\omega = 0 \qquad \text{and} \qquad *d(*\omega) = (-q, -\mathbf{J})$$

or from the tensors F and F^* using

$$\nabla' F = \left[-q\ \ -J_1\ \ -J_2\ \ -J_3\right] \qquad \text{and} \qquad \nabla' F^* = \left[0\ 0\ 0\ 0\right].$$

Also, the four vector $\mathcal{J} = (q, \mathbf{J})$ satisfies the conservation equation

$$\partial_t q + \nabla \cdot \mathbf{J} = 0.$$

The vorticity stress tensor Π was constructed from the tensor associated with vorticity W. As in Eq. (30), we construct a tensor $\mathbf{T}$ associated with F such that

$$\mathbf{T} = \frac{1}{2}\left(FF + F^* F^{*T}\right). \tag{35}$$

Thus,

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2}(-|\mathbf{L}|^2 + |\boldsymbol{w}|^2) & -(\mathbf{L}\times\boldsymbol{w})_1 & -(\mathbf{L}\times\boldsymbol{w})_2 & -(\mathbf{L}\times\boldsymbol{w})_3 \\ -(\mathbf{L}\times\boldsymbol{w})_1 & \pi_{11} & \pi_{12} & \pi_{13} \\ -(\mathbf{L}\times\boldsymbol{w})_2 & \pi_{21} & \pi_{22} & \pi_{23} \\ -(\mathbf{L}\times\boldsymbol{w})_3 & \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}, \tag{36}$$

where $\pi_{ij} = -(L_iL_j - \frac{1}{2}|L|^2\delta_{ij}) + (w_iw_j - \frac{1}{2}|\boldsymbol{w}|^2\delta_{ij})$. $\mathbf{T}$ is traceless as well.

This four-dimensional fluid dynamic stress–energy tensor $\mathbf{T}$ is analogous to the electromagnetic stress–energy tensor $\mathcal{T}$ in electromagnetic theory. As $\mathcal{T}$ is associated with the Maxwell stress tensor, $\mathbf{T}$ is associated with the vorticity stress tensor given in Eq. (29).

The electromagnetic tensor $\mathcal{F}$ is a second-rank antisymmetric tensor field that describes the electromagnetic field and the forces that would act on a charged test particle at a given location with a given velocity. The electromagnetic stress–energy tensor $\mathcal{T}$ is a symmetric second rank tensor that describes the energy and momentum carried by the electromagnetic field.

The term $-(\mathbf{L}\times\boldsymbol{w}) = \mathcal{P}$ can be seen as the Poynting vector, which describes the energy flux. The term $\frac{1}{2}(-|\mathbf{L}|^2 + |\boldsymbol{w}|^2) = \tilde{\boldsymbol{\xi}}$ represents the fluid dynamic energy density. From the fluid dynamic stress energy tensor $\mathbf{T}$, we can derive the four dimensional energy conservation laws. Evaluating $\nabla'\mathbf{T}$, we get

$$\partial_t\xi + \nabla\cdot\mathcal{P} = \mathbf{J}\cdot\mathbf{L}, \tag{37}$$

$$\begin{aligned} \partial_t\mathcal{P}_1 + \partial_i\pi_{1i} &= (\mathbf{J}\times\boldsymbol{w})_1, \\ \partial_t\mathcal{P}_2 + \partial_i\pi_{2i} &= (\mathbf{J}\times\boldsymbol{w})_2, \\ \partial_t\mathcal{P}_3 + \partial_i\pi_{3i} &= (\mathbf{J}\times\boldsymbol{w})_3. \end{aligned} \tag{38}$$

Thus, we have $\nabla'\mathbf{T} = [\mathbf{J}\cdot\mathbf{L}(\mathbf{J}\times\boldsymbol{w})_1(\mathbf{J}\times\boldsymbol{w})_2(\mathbf{J}\times\boldsymbol{w})_3]$. Equation (37) is the Poynting theorem and it describes the transfer of the energy momentum or the energy dissipation rate. The term $\mathbf{J}\cdot\mathbf{L}$ describes the rate at which the energy density is produced. The set of three equations in (38) together can be expressed as

$$\partial_t\mathcal{P} + \nabla\cdot\pi = \mathbf{J}\times\boldsymbol{w}, \tag{39}$$

which can be considered as the *fluid dynamical Lorentz force law*. If $\mathcal{J} = [q\ J_1\ J_2\ J_3]^T$, then

$$F\mathcal{J} = [\mathbf{J}\cdot\mathbf{L}\ \ -qL_1 + (\mathbf{J}\times\boldsymbol{w})_1\ \ -qL_2 + (\mathbf{J}\times\boldsymbol{w})_2\ \ -qL_1 + (\mathbf{J}\times\boldsymbol{w})_3]^T.$$

The four vector of the form $f_l = (\mathbf{J}\cdot\mathbf{L}, -q\mathbf{L} + (\mathbf{J}\times\boldsymbol{w}))$ can be considered as the four dimensional force analogous to the Lorentz force.

Thus, we have derived the fluid dynamical analogs of the Poynting theorem and the Lorentz force. Scofield and Huq[19] have derived similar results in the context of geometrodynamical theory of fluids in Minkowski space time. However, in this paper, we have derived the results that can be applied to analyze the stress–energy propagation and dissipation based on Navier–Stokes' theory in Euclidean space time.

A. A geometric algebraic approach to the Poynting theorem and Lorentz force

Let

$$\boldsymbol{F} = \nabla\boldsymbol{W} = \nabla\cdot\boldsymbol{W} + \nabla\wedge\boldsymbol{W}.$$

Now,

$$\begin{aligned} \nabla\cdot\boldsymbol{W} &= -(\nabla\cdot\mathbf{L})e_0 + (\partial_tL_1 - (\nabla\times\boldsymbol{w})_1)e_1 \\ &\quad + (\partial_tL_2 - (\nabla\times\boldsymbol{w})_2)e_2 + (\partial_tL_3 - (\nabla\times\boldsymbol{w})_3)e_3 \\ &= -qe_0 - J_1e_1 - J_2e_2 - J_3e_3 \\ &= (-q, -\mathbf{J}) = -\mathcal{J}, \end{aligned} \tag{40}$$

where q and $\mathbf{J}$ are the hydrodynamic charge density and the hydrodynamic current as defined in Sec. X. For ideal and viscous flows, it is

$$\begin{aligned} \nabla\wedge\boldsymbol{W} &= (\partial_tw_1 - (\nabla\times L)_1)e_0\wedge e_2\wedge e_3 + (\partial_tw_2 - (\nabla\times L)_2) \\ &\quad \times e_0\wedge e_3\wedge e_1 + (\partial_tw_3 - (\nabla\times L)_3)e_0\wedge e_1\wedge e_2 \\ &\quad + (\nabla\cdot\boldsymbol{w})e_1\wedge e_2\wedge e_3 \\ &= 0. \end{aligned}$$

Here, $\nabla\cdot\boldsymbol{W} = (-q, -\mathbf{J}) = -\mathcal{J}$ is a four-vector and $\nabla\wedge\boldsymbol{W} = 0$ is the trivector part of $\boldsymbol{F}$. The above four vector $\mathcal{J}$ satisfies the conservation equation

$$\partial_tq + \nabla\cdot\mathbf{J} = 0$$

for both ideal and viscous flows. Thus, the governing equations of the vorticity and the Lamb vector as derived by Marmanis[15] or otherwise termed the fluid Maxwell equations (as in Sec. X) can be extracted from a single equation

$$\boldsymbol{F} = \nabla\boldsymbol{W} = -\mathcal{J}. \tag{41}$$

Since $\boldsymbol{W} = \nabla\boldsymbol{V}$,

$$\boldsymbol{F} = \nabla^2\boldsymbol{V} = -\mathcal{J}$$

can be considered as the wave equation for $\boldsymbol{V}$.

B. Fluid dynamic stress-energy tensor

Define $T_i = \frac{1}{2}\tilde{\boldsymbol{W}}e_i\boldsymbol{W}$, where $\tilde{\boldsymbol{W}}$ is the reverse of $\boldsymbol{W}$. Since $\boldsymbol{W}$ is a bivector, we have $\tilde{\boldsymbol{W}} = -\boldsymbol{W}$. Thus, we get

$$T_0 = \frac{1}{2}(-|\mathbf{L}|^2 + |\boldsymbol{w}|^2)e_0 + (\boldsymbol{w}\times\mathbf{L})_1e_1 + (\boldsymbol{w}\times\mathbf{L})_2e_2 + (\boldsymbol{w}\times\mathbf{L})_3e_3,$$

$$\begin{aligned} T_1 &= (\boldsymbol{w}\times\mathbf{L})_1e_0 + \frac{1}{2}[-(L_1^2 - L_2^2 - L_3^2) + (w_1^2 - w_2^2 - w_3^2)]e_1 \\ &\quad + (-L_1L_2 + w_1w_2)e_2 + (-L_1L_3 + w_1w_3)e_3, \\ T_2 &= (\boldsymbol{w}\times\mathbf{L})_2e_0 + (-L_1L_2 + w_1w_2)e_1 + \frac{1}{2}[-(L_2^2 - L_1^2 - L_3^2) \\ &\quad + (w_2^2 - w_1^2 - w_3^2)]e_2 + (-L_2L_3 + w_2w_3)e_3, \end{aligned}$$

and

$$\begin{aligned} T_3 &= (\boldsymbol{w}\times\mathbf{L})_3e_0 + (-L_1L_3 + w_1w_3)e_1 + (-L_2L_3 + w_2w_3)e_2 \\ &\quad + \frac{1}{2}[-(L_3^2 - L_1^2 - L_2^2) + (w_3^2 - w_1^2 - w_2^2)]e_3. \end{aligned}$$

Here, $T_0 = \frac{1}{2}(-|\mathbf{L}|^2 + |\boldsymbol{w}|^2)e_0 + \mathcal{P}_1 e_1 + \mathcal{P}_2 e_2 + \mathcal{P}_3 e_3$ is the four dimensional Poynting vector where $\mathcal{P} = (\boldsymbol{w} \times \mathbf{L}) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$.

The elements $\mathbf{T} = [T_{ij}]$ in the fluid dynamic stress–energy tensor as given in Eq. (36) can be obtained from the scalar part of the geometric product $T_i e_j$ or we have $T_{ij} = \frac{1}{2}(\tilde{\boldsymbol{W}} e_i \boldsymbol{W} e_j)_S$, where T_{ij} is the ijth entry of the stress–energy tensor $\mathbf{T}$.

Consider

$$\boldsymbol{W}\mathcal{J} = \boldsymbol{W} \cdot \mathcal{J} + \boldsymbol{W} \wedge \mathcal{J}, \tag{42}$$

$$\begin{aligned}\boldsymbol{W} \cdot \mathcal{J} &= (L_1 J_1 + L_2 J_2 + L_3 J_3)e_0 + (-qL_1 + (\mathbf{J} \times \boldsymbol{w})_1)e_1 \\ &\quad + (-q\mathbf{L}_2 + (\mathbf{J} \times \boldsymbol{w})_2)e_2 + (-qL_3 + (\mathbf{J} \times \boldsymbol{w})_3)e_3, \\ &= [\mathbf{L} \cdot \mathbf{J}, -q\mathbf{L} + \mathbf{J} \times \boldsymbol{w}],\end{aligned} \tag{43}$$

and

$$\begin{aligned}\boldsymbol{W} \wedge \mathcal{J} &= (w_1 J_1 + w_2 J_2 + w_3 J_3)e_1 e_2 e_3 + (qw_1 + (\mathbf{L} \times \mathbf{J})_1)e_0 e_2 e_3 \\ &\quad + (qw_2 + (\mathbf{L} \times \mathbf{J})_2)e_0 e_3 e_1 + (qw_3 + (\mathbf{L} \times \mathbf{J})_3)e_0 e_1 e_2, \\ &= I[-\boldsymbol{w} \cdot \mathbf{J}, -(-q\boldsymbol{w} + \mathbf{J} \times \mathbf{L})].\end{aligned} \tag{44}$$

Therefore,

$$\frac{1}{2}(\boldsymbol{W}\mathcal{J} + \mathcal{J}\boldsymbol{W}) = \boldsymbol{W} \cdot \mathcal{J} = [\mathbf{L} \cdot \mathbf{J}, -qL + \mathbf{J} \times \boldsymbol{w}]$$

can be considered as the four dimensional force analogous to the Lorentz force in fluid dynamics. Hence, the vector part of the geometric product $\boldsymbol{W}\mathcal{J}$ yields the fluid dynamic Lorentz force. Here, we have considered only the vector part of the product $\boldsymbol{W}\mathcal{J}$. The physical and the geometric significance of the trivector part that was obtained along with the vector part needs to be further investigated.

XI. CONCLUSION

Drobot and Rybarski introduced and defined four-vectors in a Euclidean four dimensional space and developed a new variational principle for barotropic flows.[20] George Mathew and Vedan extended this variational principle to non-barotropic flows, and they derived the conservation laws using Noether's theorems.[21–23] The same were used by Geetha, Joseph, and Vedan to derive the law of conservation of potential vorticity as an application of Noether's second theorem.[24] Geetha and Vedan extended this to the study of stability of flows using Arnold's method.[25]

In this paper, we have applied the geometric algebraic methods to four dimensional flows in the Euclidean space time manifold rather than the conventional Minkowski space time manifold used elsewhere. The introduction of four-dimensional velocity vector with the Bernoulli energy function ε as the time component for barotropic flows and the generalized Bernoulli function $\tilde{f}$ in the case of non-barotropic flows in this context is a novel approach. We find that certain properties and results pertaining to the field of fluid dynamics can be obtained from the inner and outer product components of the geometric product. Most common quantities such as vorticity, helicity, and parity appear as multi vectors, i.e., sum of scalars, vectors, bivectors, trivectors, and pseudo scalars. We also find that the fluid Maxwell's equations can be extracted from a single equation. This idea of reducing the number of equations for the flow description is another major feature of this paper. The fluid dynamical stress–energy tensor, Poynting theorem, and Lorentz force discussed are important in the turbulent flow studies.[19] The analogy of the fundamental equations between two completely different disciplines in physics such as fluid dynamics and electrodynamics is very intriguing and gives a hint for an underlying common framework in nature. It also may be noted that on applying the geometric product to physical objects, we are able to extract additional quantities such as the vector quantities obtained in (22) and (24), the scalar quantities obtained in (25) and (27), and the trivector obtained in (42). However, we have not been able to make full advantage of the geometric algebraic approach as the physical and geometrical significance of these quantities obtained as byproducts of the geometric product need to be explored, paving way to future research in this area.

NOMENCLATURE

V	velocity vector in three-dimensional space
$\tilde{\boldsymbol{v}}$	generalized velocity vector in three-dimensional space
P	pressure
ρ	mass density
Φ	potential for the body forces
N	kinematic viscosity
ε	Bernoulli function
$\boldsymbol{w}$	vorticity vector in three-dimensional space
$\tilde{\boldsymbol{w}}$	generalized vorticity vector in $\mathbb{R}^3$
$\mathbb{R}^n$	Euclidean n-dimensional space
$\mathcal{V}$	velocity four vector (f, v)
σ	velocity one form in $\mathbb{R}^4$
ω	vorticity two form in $\mathbb{R}^4$
κ	parity four form
τ	thermasy
L	$\partial_t v + \nabla\varepsilon$
$d\mathbf{x}$	line element
dV	volume element
ds	surface element
S	specific entropy
Π	fluid Maxwell matrix tensor
F	vorticity field matrix tensor
$\boldsymbol{T}$	fluid dynamic stress energy matrix tensor
W	matrix tensor associated with vorticity
h	helicity density
η	helicity three form in $\mathbb{R}^4$
I	specific enthalpy
T	absolute temperature
q	hydrodynamic charge density
J	hydrodynamic current vector in $\mathbb{R}^3$
$\mathcal{J}$	hydrodynamic current density four vector (q, J) in $\mathbb{R}^4$
$\mathcal{G}_4$	geometric algebra of Euclidean space $\mathbb{R}^4$
$\mathbf{V}$	velocity four vector in $\mathcal{G}_4$
$\tilde{\mathbf{V}}$	generalized velocity four vector in $\mathcal{G}_4$
∇	vector derivative in $\mathcal{G}_4$
$\boldsymbol{W}$	vorticity bivector in $\mathcal{G}_4$
$\tilde{\boldsymbol{W}}$	generalized vorticity bivector in $\mathcal{G}_4$
$\mathbf{H}$	multivector associated with helicity in $\mathbb{R}^4$
$\tilde{\mathbf{H}}$	multivector associated with helicity for non-barotropic flows in $\mathbb{R}^4$
$\mathcal{P}$	fluid dynamical Poynting vector
e_i	basis vectors in $\mathbb{R}^4$

DATA AVAILABILITY

Data openly available in a public repository that issues datasets with DOIs. The data that support the findings of this study are available within the article.

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